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The quantum spectral transform method for the one- and two-component nonlinear Schrödinger model

Christian Elphick†

Departamento de Física, Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 5487, Santiago, Chile

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Abstract. The quantum inverse scattering method is used for the study of field theories with quartic-type interaction in 1+1 dimensions. Appropriate Lax pairs are constructed, and the corresponding Heisenberg fields are recovered from the quantised scattering data.

1. Introduction

The quantum version of the one-component nonlinear Schrödinger model is defined by the Hamiltonian

$$H=\int \mathrm{d}x\,(q_x^{\dagger}q_x+mq^{\dagger}q+cq^{\dagger}q^{\dagger}qq) \qquad c>0 \tag{1}$$

where q is a non-relativistic boson field with equal time commutation relation

$$[q^{\dagger}(x), q(y)]=\delta(x-y) \tag{2}$$

and with the weak asymptotic behaviour, as $|x|\to\infty$,

$$q(x)\to 0. \tag{3}$$

The dynamics of q is given by the so-called nonlinear Schrödinger equation (NLS)

$$\mathrm{i}q_t=-q_{xx}+mq+2cq^{\dagger}qq. \tag{4}$$

The Hamiltonian (1) describes a one-dimensional Bose gas interacting via a δ-function potential. We will consider, in both the one- and two-component models, the repulsive case $c>0$, for which the problem of interest is to determine the Green functions for a finite density ground state.

In the quantum case the inverse scattering method has been used by Faddeev and Sklyanin (1978), Sklyanin (1979) and Thacker and Wilkinson (1979) to give a complete description of the spectrum and the eigenfunctions of the Hamiltonian (1).

The inverse problem of recovering the Heisenberg field from the quantised reflection coefficient for the one-component NLS has been formulated in terms of an integral equation of Levitan–Gelfand type by Thacker and Wilkinson (1979), Göckeler (1981) and Smirnov (1982), and recently in terms of an ordinary differential equation by Lund (1982), via a lattice method (Izergin and Korepin 1981, Smirnov 1982).

† On leave from Department of Physics, Princeton University, Princeton, New Jersey 08544, USA.

In this paper we present a different approach to that in the works listed above to recover the Heisenberg fields from the quantised scattering data for both the one- and two-component NLS models.

2. One-component NLS model

We consider the normal operator version of the Zakharov–Shabat (1971) eigenvalue problem

$$L\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}\mathrm{i}(1+p)\psi_{ix}+q^\dagger\psi_2\\ \mathrm{i}(1-p)\psi_{2x}+\psi_1 q\end{pmatrix}=\lambda\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\qquad c=\frac{1}{p^2-1}\tag{5}$$

and consider also the following operator A, defined by its action on the space of eigenfunctions of L,

$$A\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}=\begin{pmatrix}p\psi_{1xx}-(1+p)^{-1}(q^\dagger\psi_1 q+(m/2c)\psi_2)-\mathrm{i}q_x^\dagger\psi_2\\ p\psi_{2xx}+(1-p)^{-1}(q^\dagger\psi_2 q+(m/2c)\psi_1)+\mathrm{i}\psi_1 q_x\end{pmatrix}.\tag{6}$$

The key point is that the eigenfunctions of L satisfy the commutation relations (which follow from (2) and (5))

$$[\psi_1(x),q(x)]=[\psi_2(x),q^\dagger(x)]=0.\tag{7}$$

Using (7) one discovers that the NLS equation is equivalent to

$$L_t=\mathrm{i}[L,A].\tag{8}$$

Relation (8) and the hermiticity of A guarantee that A generates an isospectral flow for L. In this connection the eigenfunctions of L satisfy

$$\mathrm{i}\psi_t=A\psi+f(L)\psi\qquad \psi\equiv\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\tag{9}$$

where f may be chosen from considerations of convenience. If one makes the change of variables

$$\psi_1\to(1-p)^{1/2}\exp[\mathrm{i}\lambda x/(p^2-1)]\psi_2\qquad \psi_2\to(1+p)^{1/2}\exp[\mathrm{i}\lambda x/(p^2-1)]\psi_1$$

equation (5) can be rewritten in the following way:

$$\begin{aligned}&\psi_{2x}-\mathrm{i}k\psi_2=\sqrt{c}\,q^\dagger\psi_1\\ &\psi_{1x}+\mathrm{i}k\psi_1=\sqrt{c}\,\psi_2 q\end{aligned}\qquad k=\frac{\lambda p}{1-p^2}\ \text{(spectral parameter)}.\tag{10}$$

We define now, for real k, the following two sets of different solutions of (10) (not necessarily linearly dependent):

$$\psi\equiv\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\to\begin{pmatrix}0\\ 1\end{pmatrix}\mathrm{e}^{\mathrm{i}kx}\qquad \bar\psi\equiv\begin{pmatrix}\psi_2{}^\dagger\\ \psi_1{}^\dagger\end{pmatrix}\to\begin{pmatrix}1\\ 0\end{pmatrix}\mathrm{e}^{-\mathrm{i}kx}\qquad x\to+\infty\tag{11a}$$

$$\phi\equiv\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}\to\begin{pmatrix}1\\ 0\end{pmatrix}\mathrm{e}^{-\mathrm{i}kx}\qquad \bar\phi\equiv\begin{pmatrix}\phi_2^\dagger\\ \phi_1^\dagger\end{pmatrix}\to\begin{pmatrix}0\\ 1\end{pmatrix}\mathrm{e}^{\mathrm{i}kx}\qquad x\to-\infty.\tag{11b}$$

The scattering data are defined by

$$\phi \to \begin{pmatrix} a(k)\,\mathrm{e}^{-\mathrm{i}kx} \\ b(k)\,\mathrm{e}^{\mathrm{i}kx} \end{pmatrix} \qquad x \to +\infty \tag{12}$$

and the quantised reflection and transmission coefficients by

$$R^{\dagger} = ba^{-1} \qquad T^{\dagger} = a^{-1}. \tag{13}$$

If we choose f in equation (9) in such a way that the definitions (11a) and (11b) are conserved in time, one obtains using (9), (12) and (13) that the scattering operators evolve in time in a simple way

$$R_t^{\dagger} = -\mathrm{i}(4k^2 + m)R^{\dagger} \qquad T_t^{\dagger} = 0. \tag{14}$$

2.1. *Inversion formulae*

One can show that the operators ψ, ϕ and a (resp $\bar{\psi}$, $\bar{\phi}$, $a^{\dagger}$) can be analytically continued into the upper (resp lower) half plane of the complex variable k for every x, with the following asymptotic behaviour, as $|k| \to \infty$:

$$\phi\,\mathrm{e}^{\mathrm{i}kx} \to \begin{pmatrix} 1 \\ -(\sqrt{c}/2\mathrm{i}k)q^{\dagger} \end{pmatrix} \qquad \psi\,\mathrm{e}^{-\mathrm{i}kx} \to \begin{pmatrix} (\sqrt{c}/2\mathrm{i}k)q \\ 1 \end{pmatrix} \qquad a \to 1. \tag{15}$$

In the quantum case, the main technical problem is that the relation

$$\phi = a\bar{\psi} + b\psi \tag{16}$$

is no longer valid, so one cannot follow the procedure given by Deift *et al* (1980). The problem is solved by defining the quantum analogue of (16)

$$\lambda = \bar{\psi} + R^{\dagger}\psi. \tag{17}$$

One can show that the operator λ can be analytically continued into the upper half of the complex k-plane, where it has the asymptotic behaviour, as $|k| \to \infty$,

$$\mathrm{e}^{\mathrm{i}kx}\lambda \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathrm{O}(1/k). \tag{18}$$

Intergrating the expression $\psi_1\lambda_1$ (resp $\lambda_2^{\dagger}\psi_2^{\dagger}$) on a half circle of large radius in the upper (resp lower) complex k-plane, closed by the real axis, using the asymptotic behaviour (15), and (18), one gets

$$\pi\sqrt{c}q = \int \mathrm{d}k\,(\psi_2^{\dagger}R\psi_2^{\dagger} - \psi_1 R^{\dagger}\psi_1). \tag{19}$$

It is worthwhile to note that if one considers the expression $\lambda_1\psi_2$ (resp $\psi_2^{\dagger}\lambda_2^{\dagger}$), one recovers Lund's result (Lund 1982)

$$\pi\sqrt{c}q = \int \mathrm{d}k\,(\psi_2^{\dagger}\psi_2^{\dagger}R - R^{\dagger}\psi_1\psi_1).$$

Using similar reasoning, it is also possible to obtain the inversion formulae

$$2\pi\sqrt{c}q_x = \int \mathrm{d}k\,(c - 4\mathrm{i}k)(\psi_2^{\dagger}R\psi_2^{\dagger} - \psi_1 R^{\dagger}\psi_1) \tag{20a}$$

$$\tfrac{1}{2}\mathrm{i}\pi c q^{\dagger}(x)q(y) = \int \mathrm{d}k\, k\mathrm{e}^{-\mathrm{i}k(y-x)}(\psi_1^{\dagger}(x,k)R\psi_2^{\dagger}(y,k) - \psi_2(x,k)R^{\dagger}\psi_1(y,k))$$
$$= \int \mathrm{d}k\, k\mathrm{e}^{-\mathrm{i}k(y-x)}(\psi_1^{\dagger}(x)\psi_2^{\dagger}(y)R - R^{\dagger}\psi_2(x)\psi_1(y)). \tag{20b}$$

If one takes $y=x$ in (20*b*) one recovers Lund's result (Lund 1982). Moreover, the infinite number of conserved quantities for the model also admit inversion formulae

$$a_{n+1} = \frac{1}{n!\mathrm{i}\pi}\int \mathrm{d}k\, (\psi_2^{\dagger}\psi_2 + R^{\dagger}\psi_1\psi_2)(-)^{n+1}k^n \frac{\mathrm{d}^n}{\mathrm{d}k^n}(k^n \ln a) \qquad n=0,1,\ldots. \tag{21}$$

Finally, the inverse method is completed if one substitutes (19) into (10), then solves (10) together with the definition (11), substituting the solution back into (19) to get q as a function of the quantised reflection coefficient.

3. Two-component NLS model

The model is defined by the Hamiltonian

$$H = \int \mathrm{d}x\, (q_x^{\dagger}q_x + mq^{\dagger}q + cq^{\dagger}q^{\dagger}\Lambda qq) \qquad c>0 \tag{22}$$

where

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \qquad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & \sigma^2 \end{pmatrix} \tag{23}$$

together with the equal time commutation relations

$$[q_i^{\dagger}(x), q_j(y)] = \delta_{ij}\delta(x-y) \tag{24}$$

and with the weak asymptotic behaviour, as $|x|\to\infty$,

$$q_1 \to 0 \qquad q_2 \to 0. \tag{25}$$

The equations of motion read

$$\mathrm{i}q_{1t} = -q_{1xx} + mq_1 + 2c(q_1^{\dagger}q_1 + \sigma^2 q_2^{\dagger}q_2)q_1 \tag{26}$$
$$\mathrm{i}q_{2t} = -q_{2xx} + mq_2 + 2c(q_1^{\dagger}q_1 + \sigma^2 q_2^{\dagger}q_2)q_2.$$

The generalisation of the associated problem (5), (6) corresponds to

$$L\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_2 \end{Bmatrix} = \begin{Bmatrix} \mathrm{i}(1+p)\psi_{1x} + q_1^{\dagger}\psi_3 \\ \mathrm{i}(1+p)\psi_{2x} + \sigma q_2^{\dagger}\psi_3 \\ \mathrm{i}(1-p)\psi_{3x} + \psi_1 q_1 + \sigma\psi_2 q_2 \end{Bmatrix} = \lambda \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix} \qquad c = \frac{1}{p^2-1} \tag{27}$$

$$A\begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Bmatrix} = \begin{Bmatrix} p\psi_{1xx} - (1+p)^{-1}[q_1^{\dagger}\psi_1 q_1 + \sigma q_1^{\dagger}\psi_2 q_2 + (m/2c)(\psi_1+\psi_2)] - \mathrm{i}q_{1x}^{\dagger}\psi_3 \\ p\psi_{2xx} - (1+p)^{-1}(\sigma q_2^{\dagger}\psi_1 q_1 + \sigma q_2^{\dagger}\psi_2 q_1 + (m/2c)(\psi_1+\psi_2)) - \mathrm{i}\sigma q_{2x}^{\dagger}\psi_3 \\ p\psi_{3xx} + (1+p)^{-1}(q_1^{\dagger}\psi_3 q_1 + \sigma^2 q_2^{\dagger}\psi_3 q_2 + (m/2c)\psi_3) + \mathrm{i}\psi_1 q_{1x} + \mathrm{i}\sigma\psi_2 q_2 \end{Bmatrix}, \tag{28}$$

As in the one-component case, the key point is the commutation relations (which follow from (24) and (27))

$$[\psi_3(x), q_1(x)] = [\psi_3(x), q_2(x)] = 0 \qquad [\psi_i(x), q_i^\dagger(x)] = 0 \qquad i = 1, 2. \tag{29}$$

Using (29), one obtains that (26) is equivalent to

$$L_t = \mathrm{i}[L, A] \tag{30}$$

and that the eigenfunctions of L evolve in time according to

$$\mathrm{i}\psi_t = A\psi + f(L)\psi. \tag{31}$$

We consider now the L-eigenvalue problem (27), and make the change of variables

$$\psi_1 \to (1-p)^{1/2} \exp[\mathrm{i}\lambda x/(p^2-1)]\psi_3 \qquad \psi_2 \to (1-p)^{1/2} \exp[\mathrm{i}\lambda x/(p^2-1)]\psi_2$$
$$\psi_3 \to (1+p)^{1/2} \exp[\mathrm{i}\lambda x/(p^2-1)]\psi_1. \tag{32}$$

Equation (27) can be rewritten in the following way:

$$\psi_{1x} + \mathrm{i}k\psi_1 = \psi_3\hat{q}_1 + \psi_2\hat{q}_2 \qquad \psi_{2x} - \mathrm{i}k\psi_2 = \hat{q}_2^\dagger\psi_1 \qquad \psi_{3x} - \mathrm{i}k\psi_3 = \hat{q}_1^\dagger\psi_1 \tag{33}$$

$$\hat{q}_1 = \sqrt{c}\,q_1 \qquad \hat{q}_2 = \sigma\sqrt{c}\,q_2 \qquad k = \lambda p/(1-p^2). \tag{34}$$

We define the following sets of different solutions of (33) (not necessarily linearly dependent):

$$\psi^{(1)} \to \begin{pmatrix}1\\0\\0\end{pmatrix} \mathrm{e}^{-\mathrm{i}kx} \qquad \psi^{(2)} \to \begin{pmatrix}0\\1\\0\end{pmatrix} \mathrm{e}^{\mathrm{i}kx} \qquad \psi^{(3)} \to \begin{pmatrix}0\\0\\1\end{pmatrix} \mathrm{e}^{\mathrm{i}kx} \qquad x \to +\infty$$

$$\phi^{(1)} \to \begin{pmatrix}1\\0\\0\end{pmatrix} \mathrm{e}^{-\mathrm{i}kx} \qquad \phi^{(2)} \to \begin{pmatrix}0\\1\\0\end{pmatrix} \mathrm{e}^{\mathrm{i}kx} \qquad \phi^{(3)} \to \begin{pmatrix}0\\0\\1\end{pmatrix} \mathrm{e}^{\mathrm{i}kx} \qquad x \to -\infty. \tag{35}$$

The generalised scattering data are defined by the asymptotic behaviour of (ϕ^1, ϕ^2, ϕ^3) as $x \to +\infty$

$$\phi_1^{(j)} \to a_j(k)\,\mathrm{e}^{-\mathrm{i}kx} \qquad \phi_2^{(j)} \to b_{j2}(k)\,\mathrm{e}^{\mathrm{i}kx} \qquad \phi_3^{(j)} \to b_{j3}(k)\,\mathrm{e}^{\mathrm{i}kx} \qquad j = 1, 2, 3. \tag{36}$$

As in the one-component case, we choose f in (31) such that the definition (35) is conserved in time. Having done this, and using (31), one gets

$$\begin{gathered} a_{1t} = b_{22t} = b_{23t} = b_{32t} = b_{33t} = 0 \\ b_{12t} = -\mathrm{i}(4k^2+m)b_{12} \qquad b_{13t} = -\mathrm{i}(4k^2+m)b_{13} \\ a_{2t} = \mathrm{i}(4k^2+m)a_2 \qquad a_{3t} = \mathrm{i}(4k^2+m)a_3. \end{gathered} \tag{37}$$

It is also convenient to define the operators

$$\begin{aligned} \Delta^{(1)} &= \psi^{(1)} + b_{12}a_1^{-1}\psi^{(2)} + b_{13}a_1^{-1}\psi^{(3)} \\ \Delta^{(2)} &= \psi^{(2)} + \psi^{(1)}b_{22}^{-1}a_2 + \psi^{(3)}b_{22}^{-1}b_{23} \\ \Delta^{(3)} &= \psi^{(3)} + \psi^{(1)}b_{33}^{-1}a_3 + \psi^{(2)}b_{33}^{-1}b_{32}. \end{aligned} \tag{38}$$

One can show that the operators $\phi^{(1)}$, $\psi^{(2)}$, $\psi^{(3)}$, a_1, $\Delta^{(1)}$ (resp $\psi^{(1)}$, $\phi^{(2)}$, $\phi^{(3)}$, b_{22}, b_{32}, b_{23}, b_{33}, $\Delta^{(2)}$, $\Delta^{(3)}$) can be analytically continued in the upper (resp lower) complex

k-plane for every x, where they have the asymptotic behaviour, as $|k|\to\infty$,

$$\begin{aligned}
e^{ikx}\psi^{(1)} &\to \begin{Bmatrix} 1 \\ -\hat{q}_2^\dagger/2ik \\ -\hat{q}_1^\dagger/2ik \end{Bmatrix} \leftarrow e^{ikx}\Delta^{(1)} \\
e^{-ikx}\psi^{(2)} &\to \begin{Bmatrix} \hat{q}_2/2ik \\ 1 \\ 0 \end{Bmatrix} \leftarrow e^{-ikx}\Delta^{(2)} \\
e^{-ikx}\psi^{(3)} &\to \begin{Bmatrix} \hat{q}_1/2ik \\ 0 \\ 1 \end{Bmatrix} \leftarrow e^{-ikx}\Delta^{(3)}.
\end{aligned} \tag{39}$$

3.1. *Inversion formulae*

We proceed now in a similar way to the one-component case. For instance, we outline below the procedure to obtain the Heisenberg fields $\hat{q}_1$, $\hat{q}_2$. We integrate the expression $\psi_1^{(3)}\Delta_1^{(1)}$ (resp $\Delta_1^{(3)}\psi_1^{(1)}$) on a half circle of large radius in the upper (resp lower) complex k-plane, closed by the real axis, using the aymptotic behaviour (39), and the analyticity properties; finally, subtracting the two contour integrals, one gets

$$\pi\hat{q}_1 = \int dk\, (\psi_1^{(1)} b_{33}^{-1} a_3 \psi_1^{(1)} + \psi_1^{(2)} b_{33}^{-1} b_{32} \psi_1^{(1)} - \psi_1^{(3)} b_{13} a_1^{-1} \psi_1^{(3)} - \psi_1^{(3)} b_{12} a_1^{-1} \psi_1^{(2)}). \tag{40}$$

For the field $\hat{q}_2$, one has to consider the expression $\psi_1^{(2)}\Delta_1^{(1)}$ (resp $\Delta_1^{(2)}\psi_1^{(1)}$); the result is

$$\pi\hat{q}_2 = \int dk\, (\psi_1^{(3)} b_{22}^{-1} b_{23} \psi_1^{(1)} + \psi_1^{(1)} b_{22}^{-1} a_2 \psi_1^{(1)} - \psi_1^{(2)} b_{13} a_1^{-1} \psi_1^{(3)} - \psi_1^{(2)} b_{12} a_1^{-1} \psi_1^{(2)}). \tag{41}$$

Following a similar procedure, one obtains the more useful formulae

$$\begin{aligned}
\pi\hat{q}_1 &= \int dx\, (\psi_3^{(3)\dagger} R_1 \psi_1^{(1)} + \psi_3^{(2)\dagger} R_2 \psi_1^{(1)} - \psi_3^{(1)\dagger} R_2^\dagger \psi_1^{(3)} - \psi_3^{(1)\dagger} R_1^\dagger \psi_1^{(2)}) \\
\pi\hat{q}_2 &= \int dx\, (\psi_2^{(3)\dagger} R_1 \psi_1^{(1)} + \psi_2^{(2)\dagger} R_2 \psi_1^{(1)} - \psi_2^{(1)\dagger} R_2^\dagger \psi_1^{(3)} - \psi_2^{(1)\dagger} R_1^\dagger \psi_1^{(2)})
\end{aligned} \tag{42}$$

where the reflection coefficients $R_1^\dagger$ and $R_2^\dagger$ are defined by

$$R_1^\dagger = b_{13} a_1^{-1} \qquad R_2^\dagger = b_{12} a_1^{-1}. \tag{43}$$

The connection with the one-component case formula (19) is now evident. Using similar reasoning it is also possible to show

$$\begin{aligned}
&\tfrac{1}{2} c i \pi \hat{q}_1^\dagger(x) \hat{q}_1(y) \\
&\quad= \int dk\, k e^{ik(y-x)} (\psi_3^{(1)}(x)\psi_1^{(1)}(y) b_{33}^{-1} a_3 + \psi_3^{(1)}(x)\psi_1^{(2)}(y) b_{33}^{-1} b_{32} \\
&\qquad - b_{13} a_1^{-1} \psi_3^{(3)}(x)\psi_1^{(2)}(y) - b_{12} a_1^{-1} \psi_3^{(2)}(x)\psi_1^{(2)}(y)) \\
&\quad= \int dk\, k e^{ik(y-x)} (\psi_3^{(1)}(x)\psi_1^{(2)\dagger}(y) R_1 + \psi_3^{(1)}(x)\psi_1^{(3)\dagger}(y) R_2 \\
&\qquad - R_1^\dagger \psi_1^{(2)}(x)\psi_3^{(1)\dagger}(y) - R_2^\dagger \psi_1^{(3)}(x)\psi_3^{(1)\dagger}(y))
\end{aligned}$$

$$\tfrac{1}{2}ci\pi\hat{q}_2^\dagger(x)\hat{q}_2(y)$$

$$= \int dk\, k\, e^{ik(y-x)}(\psi_2^{(1)}(x)\psi_1^{(3)}(y)b_{22}^{-1}b_{23}+\psi_2^{(1)}(x)\psi_1^{(1)}(y)b_{22}^{-1}a_2$$

$$-b_{13}a_1^{-1}\psi_2^{(3)}(x)\psi_1^{(2)}(y)-b_{12}a_1^{-1}\psi_2^{(2)}(x)\psi_1^{(2)}(y))$$

$$= \int dk\, k\, e^{ik(y-x)}(\psi_2^{(1)}(x)\psi_2^{(2)\dagger}(y)R_1+\psi_2^{(1)}(x)\psi_2^{(3)\dagger}(y)R_2$$

$$-R_1^\dagger\psi_2^{(2)}(x)\psi_2^{(1)\dagger}(y)-R_2^\dagger\psi_2^{(3)}(x)\psi_2^{(1)\dagger}(y)). \tag{44}$$

4. General remarks

(1) The fact that one has to introduce (17) (resp (38)) is precisely the argument that the sets of solutions (11) (resp (35)) are not linearly dependent when these are operators. If one takes them as c-numbers one recovers the usual classical expressions.

(2) By using the inversion formulae (42) one can prove that

$$q_2^\dagger(x_1)\ldots q_2^\dagger(x_n)q_1^\dagger(y_1)\ldots q_1^\dagger(y_m)|0\rangle = R_2^\dagger(x_1)\ldots R_2^\dagger(x_n)R_1^\dagger(y_1)\ldots R_1^\dagger(y_n)|0\rangle \tag{45}$$

where

$$R_2^\dagger(x)=(\sqrt{c}\sigma 2\pi)^{-1}\int dk\, e^{ikx}R_2^\dagger(k/2) \qquad R_1^\dagger(x)=(\sqrt{c}2\pi)^{-1}\int dk\, e^{ikx}R_1^\dagger(k/2)$$

and $x_1\geqslant\ldots\geqslant x_n$; $y_1\geqslant\ldots\geqslant y_m$.

On the other hand, using (37) we obtain that the eigenstates of (22) (up to an irrelevant phase) are given by

$$R_2^\dagger(k_1/2)\ldots R_2^\dagger(k_n/2)R_1^\dagger(p_1/2)\ldots R_1^\dagger(p_m/2)|0\rangle. \tag{46}$$

Finally, using (45) it is possible to show that the wavefunction of the state (46) turns out to coincide with the wavefunction obtained by Bethe ansatz (Öttinger 1981).

5. Conclusions

In this work we have been able to construct the Heisenberg fields in terms of the quantised scattering data, for the one- and two-component nonlinear Schrödinger theory. We have also derived in the one-component case inversion formulae for q_x, q_{xx} and q_t, which can be useful to study the short-distance and short-time behaviours of the Green function of the model for finite values of the coupling constant. Of special interest is formula (A2.2), since this will let us avoid using the complicated time evolution of the quantised Jost functions given by (6) and (9).

At this stage the last remarks are only a conjecture, but progress is being carried out in this direction, which we hope to report in a subsequent paper.

We think that rather than formulae (19) and (20b), the crucial formulae of our formalism are (20b), (44) and (A2.3), which we hope will provide some new insight in connection with computing the propagators (n-point Green functions) of the models (one- and two-component NLS) using non-perturbative techniques.

As is seen in appendix 1, our formalism is in some way equivalent to previous formulations (integral-type), so it follows that the quantisation procedure followed (infinite support) is equivalent to the usual normal ordering procedures. At this point I would like to remark that if one wants to get normal ordered expressions with respect to the physical vacuum, one can (following Jimbo *et al* (1980)) add a term $\lambda(q^\dagger q^\dagger + qq)$ in the Hamiltonian which allows us to consider $\langle q(x)\rangle \to$ constant, as $|x| \to \infty$, analogous to the classical vacuum at finite density; since this new term affects the A operator and not the L operator, it follows that the spatial dependence of the Jost functions remains unchanged, which implies that the inversion formulae remain unchanged, too. As a last step one takes $\lambda \to 0$, to get the correct physical observables.

Despite the equivalence mentioned before, our formalism has the advantage of formulae (20*b*), (44) and (A2.3) (so related with n-point functions), which are not possible to derive following the usual formulations.

Finally, I would like to remark that using formulae (42) and (43), one can readily get Öttinger's expansion formulae for the fields q_1 and q_2 (Öttinger 1981).

Appendix 1. Connection with integral-type formulations

A1.1. One-component case

We will obtain here the relation between our formalism, which is formulated in terms of an ordinary differential equation (10), and the usual formalism based on integral equations.

To begin with, we will derive the quantum analogue of the Marchenko equation (Zakharov and Shabat (1971), for the classical case). To accomplish this we consider the following operator (which was introduced in § 2.1)

$$\lambda = \bar{\psi} + R^\dagger \psi \tag{A1.1}$$

together with the definitions

$$\psi_1(x, k) = \int_x^\infty \mathrm{d}z\, \mathrm{e}^{\mathrm{i}kz} K_1(x, z) \qquad \psi_2(x, k) = \mathrm{e}^{\mathrm{i}kz} + \int_x^\infty \mathrm{d}z\, \mathrm{e}^{\mathrm{i}kz} K_2(x, z). \tag{A1.2}$$

On replacing (A1.2) in (10), one readily gets the conditions

$$K_1(x, x) = -\tfrac{1}{2}\sqrt{c}\, q(x) \qquad K_2(x, x) = \tfrac{1}{2} c \int_x^\infty \mathrm{d}z\, q^\dagger(z) q(z). \tag{A1.3}$$

To obtain the quantised Marchenko equation we substitute (A1.2) into (A1.1) to find

$$\lambda(x, k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx} + \int_x^\infty \mathrm{d}z\, \bar{K}(x, z)\, \mathrm{e}^{-\mathrm{i}kz} + R^\dagger(k)\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}kx} + \int_x^\infty \mathrm{d}z\, K(x, z)\, \mathrm{e}^{\mathrm{i}kz}\right) \tag{A1.4}$$

where we have defined

$$K = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} \qquad \bar{K} = \begin{pmatrix} K_2^\dagger \\ K_1^\dagger \end{pmatrix}. \tag{A1.5}$$

By operating on (A1.4) with $(2\pi)^{-1} \int \mathrm{d}k\, \mathrm{e}^{\mathrm{i}ky}$, and using the analytic and asymptotic properties of λ (see § 2.1), we finally get

$$\bar{K}(x, y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} R^\dagger(x + y) + \int_x^\infty \mathrm{d}z\, R^\dagger(z + y) K(x, z) = 0 \tag{A1.6}$$

where

$$R^{\dagger}(x) = (2\pi)^{-1} \int \mathrm{d}k\, R^{\dagger}(k)\, \mathrm{e}^{\mathrm{i}kx}.$$

In order to obtain the desired relation, we need only to prove (A1.3) using our formalism. To accomplish this we replace (A1.2) in (19) to arrive at

$$\begin{aligned}\pi\sqrt{c}q(x) = 2\pi\biggl(R(2x) &+ \int_x^{\infty} \mathrm{d}v\, K_2^{\dagger}(x, v)R(x+v) + \int_x^{\infty} \mathrm{d}v\, K_2^{\dagger}(x, v)R(x+v) \\ &+ \int_x^{\infty} \mathrm{d}z\, \mathrm{d}v\, K_2^{\dagger}(x, z)K_2^{\dagger}(x, v)R(z+v) \\ &- \int_x^{\infty} \mathrm{d}z\, \mathrm{d}v\, R^{\dagger}(z+v)K_1(x, z)K_1(x, v)\biggr).\end{aligned} \tag{A1.7}$$

The reiterated use of (A1.6) allows us to prove that the contribution of the last three terms in (A1.7) vanishes identically. Thus we arrive at

$$\sqrt{c}q(x) = 2\biggl(R(2x) + \int_x^{\infty} \mathrm{d}v\, K_2^{\dagger}(x, v)R(x+v)\biggr). \tag{A1.8}$$

Using again (A1.6), we easily find that the right-hand side of (A1.8) is nothing else but $-2K_1(x, x)$, thus recovering the first of the two conditions (A1.3).

To recover the second one, we use (20*b*) together with (A1.2) and we easily get

$$\mathrm{d}K_2(x, x)/\mathrm{d}x = -\tfrac{1}{2}cq^{\dagger}(x)q(x). \tag{A1.9}$$

Integrating (A1.9), and using (A1.2) to fix the constant of integration, give us the complete recovery of (A1.3).

In this way we have established the equivalence of (19) and (20*b*) with the form (A1.6) of the inverse scattering formalism.

Finally, it is also possible to make a connection between our formalism and the Gel'fand–Levitan integral equation given by Thacker (1981). Because of the rather tedious calculations involved, we will merely outline how the formalisms are related.

With the aid of (16) and (19), it is easy to see that the quantised Jost functions satisfy

$$\begin{aligned}\psi_1(x, k)\, \mathrm{e}^{\mathrm{i}kx} &= -\frac{\sqrt{c}}{\pi\mathrm{i}} \int_x^{\infty} \mathrm{d}z\, \mathrm{d}k'\, \mathrm{e}^{\mathrm{i}kz}\psi_2(z, k)(\psi_2^{\dagger}(z, k')\hat{R}^{\dagger}(k')\psi_2^{\dagger}(z, k') \\ &\quad - \psi_1(z, k')\hat{R}(k')\psi_1(z, k')) \\ \psi_2(x, k)\, \mathrm{e}^{-\mathrm{i}kx} &= 1 - \frac{\sqrt{c}}{\pi\mathrm{i}} \int_x^{\infty} \mathrm{d}z\, \mathrm{d}k'\, \mathrm{e}^{\mathrm{i}kz}(\psi_2(z, k')\hat{R}(k')\psi_2(z, k') \\ &\quad - \psi_1^{\dagger}(z, k')\hat{R}^{\dagger}(k')\psi_1^{\dagger}(z, k'))\psi_2(z, k)\end{aligned} \tag{A1.10}$$

where $\hat{R} = (\mathrm{i}/\sqrt{c})R^{\dagger}$ (Thacker's definition of the reflection coefficient). If one iterates (A1.10), one exactly gets Thacker's Gel'fand–Levitan series for the Jost functions, and if one replaces these series in (19) one arrives at Thacker's expansion of the Heisenberg field q.

A1.2. Two-component case

Following a similar procedure to the previous one, we define a convenient generalisation of the operator (A1.1). It is not difficult to see that this operator is given by

$$\nabla = \psi^{(1)} + R_2^\dagger \psi^{(2)} + R_1^\dagger \psi^{(3)} \tag{A1.11}$$

where $R_2^\dagger$ and $R_1^\dagger$ have been defined in (43). As before, we make the following definitions:

$$\begin{aligned}
\psi^{(1)}(x,k) &= \begin{pmatrix}1\\0\\0\end{pmatrix} e^{-ikx} + \int_x^\infty dz\, K^{(1)}(x,z)\, e^{-ikz} \\
\psi^{(2)}(x,k) &= \begin{pmatrix}0\\1\\0\end{pmatrix} e^{ikx} + \int_x^\infty dz\, K^{(2)}(x,z)\, e^{ikz} \\
\psi^{(3)}(x,k) &= \begin{pmatrix}0\\0\\1\end{pmatrix} e^{ikx} + \int_x^\infty dz\, K^{(3)}(x,z)\, e^{ikz}.
\end{aligned} \tag{A1.12}$$

If one substitutes (A1.12) in (33) one arrives at the conditions

$$\hat{q}_1(x) = -\tfrac{1}{2} K_1^{(3)}(x,x) \qquad \hat{q}_2(x) = -\tfrac{1}{2} K_1^{(2)}(x,x). \tag{A1.13}$$

As in the previous case we replace (A1.12) in (A1.11), operate with $(2\pi)^{-1}\int dk\, e^{iky}$, and finally use the analytic and asymptotic properties of ∇(see (3.1)), to get the quantised Marchenko equation for the two-component NLS model

$$\begin{aligned}
K^{(1)}(x,y) &+ \begin{pmatrix}0\\1\\0\end{pmatrix} R_2^\dagger(x+y) + \begin{pmatrix}0\\0\\1\end{pmatrix} R_1^\dagger(x+y) + \int_x^\infty dz\, R_2^\dagger(z+y) K^{(2)}(x,z) \\
&+ \int_x^\infty dz\, R_1^\dagger(z+y) K^{(3)}(x,z) = 0.
\end{aligned} \tag{A1.14}$$

To prove that our formalism is equivalent to the Marchenko-type formalism, we need only to obtain (A1.13) using (42). The proof is rather lengthy and straightforward, so I will merely outline how to get it. One replaces the definition (A1.12) in (42), and then by the repeated use of (A1.14) one arrives at (A1.13).

It is also worth noting that the series expansion for the operator Jost functions obtained by using (A1.14) is the same if one substitutes (42) back into (33). In other words, our formalism also gives the Gel'fand–Levitan series expansion for the two-component NLS model.

Appendix 2. Other inversion formulae

I will outline here how to obtain some new inversion formulae, which I believe will be useful in connection with short-time behaviours and n-point Green functions.

First I will derive two new formulae, which even in the classical case have not been derived yet. By taking the spatial derivative of (20a) we get (after some simplifications)

$$\sqrt{c}q_{xx}=\pi^{-1}\int \mathrm{d}k\,(\tfrac{1}{2}c-2\mathrm{i}k)^2(\psi_2^\dagger R\psi_2^\dagger-\psi_1 R^\dagger\psi_1)+c(q^\dagger qq+qq^\dagger q). \tag{A2.1}$$

The last term of (A2.1) can be rewritten as (using the asymptotic properties of λ_2, ψ_1 as $|k|\to\infty$)

$$-\frac{1}{\sqrt{c}}\frac{8}{\pi}\int \mathrm{d}k\,k^2\,\mathrm{e}^{-\mathrm{i}kx}(\psi_1^\dagger\psi_1\psi_1+R^\dagger\psi_2\psi_1\psi_1+\psi_1R^\dagger\psi_2\psi_1+\psi_1\psi_1^\dagger\psi_1). \tag{A2.1a}$$

On replacing (A2.1a) in (A2.1) we finally arrive at the inversion formula for q_{xx}.The classical formula for q_{xx} is readily found by taking $(c/2-2\mathrm{i}k)\to(-2\mathrm{i}k)$ in the first term of (A2.1).

The second formula is for the time derivative of q. To obtain it, we use (A2.1), (A2.1a), and the equation of motion (4), and we get

$$\mathrm{i}q_t=\frac{1}{\pi}\int \mathrm{d}k[m-(\tfrac{1}{2}c-2\mathrm{i}k)^2](\psi_2^\dagger R\psi_2^\dagger-\psi_1R^\dagger\psi_1)-\frac{\mathrm{i}\sqrt{c}}{\pi}\int \mathrm{d}k\,k\,\mathrm{e}^{-\mathrm{i}kx}\psi_1. \tag{A2.2}$$

To get the classical formula for q_t we take $(c/2-2\mathrm{i}k)\to(-2\mathrm{i}k)$ in the first term of (A2.2) and do not consider the last term of (A2.2) since it arises from quantum orderings.

Finally I will merely write an inversion formula for the product of $2N$ Heisenberg fields (which can be derived following the techniques used in § 2.1). The formula is

$$\frac{\mathrm{i}c^N\pi}{2^{2N-1}}\prod_{i=1}^N q^\dagger(x_i)\prod_{i=1}^N q(y_i)=\int \mathrm{d}k\,k^{2N-1}\prod_{i=1}^N \exp[\mathrm{i}k(x_i-y_i)]$$
$$\times\left(\prod_{i=1}^N\psi_1^\dagger(x_i,k)\prod_{i=1}^N\lambda_2^\dagger(y_i,k)-\prod_{i=1}^N\lambda_2(x_i,k)\prod_{i=1}^N\psi_1(y_i,k)\right). \tag{A2.3}$$

For instance, if one takes $N=2$ in (A2.3) one explicitly gets

$$\tfrac{1}{8}\mathrm{i}c^2\pi q^\dagger(x_1)q^\dagger(x_2)q(y_1)q(y_2)=\int \mathrm{d}k\,k^3\exp[\mathrm{i}k(x_1+x_2-y_1-y_2)]$$
$$\times(\{\psi_1^\dagger(x_1,k)\psi_1^\dagger(x_2,k)[\psi_1(y_1,k)\psi_2^\dagger(y_2,k)R(k)$$
$$+\psi_2^\dagger(y_1,k)R(k)\psi_1(y_2,k)+\psi_2^\dagger(y_1,k)R(k)\psi_2^\dagger(y_2,k)R(k)]\}$$
$$-\{x_1\leftrightarrow y_2,x_2\leftrightarrow y_1\}^\dagger).$$

It is worth remarking that there is no way to get formulae like (A2.3) using other formulations of the inverse method (even in the classical case). In the quantum case (A2.3) is relevant in connection with n-point Green functions.

References

Deift P, Lund F and Trubowitz E 1980 *Commun. Math. Phys.* **74** 141–88
Faddeev L D and Sklyanin E K 1978 *Dokl. Akad. Nauk SSSR* **243** 1430–3
Göckeler M 1981 *Z. Phys.* C **11** 125–32
Izergin A G and Korepin V E 1981 *Dokl. Akad. Nauk SSSR* **259** 76–9

Jimbo M, Miwa T, Mori Y and Sato M 1980 *Physica* **1D** 80
Lund F 1982 *Phys. Lett.* **120B** 187–90
Manakov S V 1973 *Zh. Eksp. Teor. Fiz.* **65** 505–16
Öttinger H C 1981 *Physica* **107A** 423–30
Sklyanin E K 1979 *Dokl. Akad. Nauk SSSR* **244** 1337–40
Smirnov F A 1982 *Dokl. Akad. Nauk SSSR* **262** 78–81
Thacker H B 1981 *Rev. Mod. Phys.* **53** 253–85
Thacker H B and Wilkinson D 1979 *Phys. Rev.* **19** 3660–5
Zakharov V E and Shabat A B 1971 *Zh. Eksp. Teor. Fiz.* **61** 118–34